Toric data, Killing forms and complete integrability of geodesics in Sasaki-Einstein spaces $Y^{p,q}$

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Abstract

In the present paper we show that the complete list of special Killing forms on the 5-dimensional Sasaki-Einstein spaces $Y^{p,q}$ can be extracted using the symplectic potential and the classical Delzant construction. The results achieved here agree with previous ones obtained by direct computation, proving the reliability of the method which stands in fact as a general algorithm for toric Sasaki-Einstein manifolds. Finally, we discuss the integrability of geodesic motion in $Y^{p,q}$ spaces.

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1 Introduction

Symmetries of dynamical equations have always played very important role in physics. The most common meaning of the symmetry is associated to that of isometry that leaves the metric invariant. A one-parameter continuous isometry is related to Killing vectors. More general one can consider a physical system evolving in a given background and analyze the symmetries of the whole phase space of the system such that the dynamics is left invariant. Such symmetries are often referred to as hidden symmetries whose conserved quantities are of higher order in momenta.

The conserved quantities polynomial in momenta are constructed making use of Stäckel-Killing tensors which are natural generalization of the Killing vectors. There is also an antisymmetric generalization of the Killing vectors represented by Killing-Yano tensors. They play an important role in the study of complete integrability of geodesic equations and complete separation of variables for the Hamilton-Jacobi, Klein-Gordon and Dirac equations in general Kerr-NUT-(A)dS metrics.

In the last time Sasaki-Einstein geometry has become of significant interest in some modern developments in mathematics and theoretical physics [6, 23]. The Sasaki-Einstein spaces provide supersymmetric backgrounds relevant to the AdS/CFT correspondence [13].

In this paper we want to take a closer look at the special Killing forms on the 5-dimensional toric Sasaki-Einstein $Y^{p,q}$ spaces. The special Killing forms on the Sasaki-Einstein manifold $Y^{p,q}$ were firstly obtained by a direct calculation in [25]. Later, in [20], the authors show that working with foliated coordinates we can also extract the special Killing forms on a Sasaki-Einstein manifolds, developing a general alternative approach which was exemplified in the case of the five-dimensional $Y^{p,q}$ spaces, obtaining the same results as in [25]. Now we give a third approach, proving that the description of the Calabi-Yau cone $C(Y^{p,q})$ using toric data allows us to extract the special Killing forms on $Y^{p,q}$ based on a standard method. Using the complete set of special Killing forms we construct the Stäckel-Killing tensors and the corresponding conserved quantities, quadratic in momenta. Finally we investigate the integrability of the geodesic motion and show that the system is completely integrable.

The paper has the following organization: In Section 2.1 we review some well-known properties of Killing tensors. In Section 2.2 we describe the special Killing forms on Sasaki-Einstein manifolds. In Sections 2.3 and 2.4 we present the evaluation of symplectic potential and the holomorphic volume form in complex coordinates using toric data. In Section 3 we exemplify the general scheme in the case of the five dimensional Sasaki-Einstein spaces $Y^{p,q}$ and write down the symplectic and complex coordinates. In Section 4 we present the complete set of special Killing forms on $Y^{p,q}$ spaces. In Section 5 we evaluate the Stäckel-Killing tensors constructed from Killing-Yano tensors and prove the complete integrability of geodesic motion in $Y^{p,q}$ spaces. The paper ends with conclusions in Section 6.

2 Preliminaries

2.1 Killing forms and Stäckel-Killing tensors

The *Killing forms* (sometimes called *Killing-Yano tensors*) stand as a natural extension of classical Killing 1-forms (which are dual to Killing vector fields). We introduce these differentiable forms in accordance with [19]. Throughout the paper we use standard conventions: ∇ is the Levi-Civita connection with respect to the metric g, X^* is the 1-form dual to the vector field X, $\lrcorner$ is the operator dual to the wedge product.

Definition 1. *If (M,g) is a Riemannian manifold, then a Killing form of rank p is a p-form ψ which have to satisfy the equation*

$$\nabla_X \psi = \frac{1}{p+1} X \lrcorner d\psi \,, \tag{1}$$

for any vector field X on M.

The corresponding equation of (1) in component notation becomes

$$\nabla_{(j}\psi_{i_1)i_2\dots i_p} = 0 \,.$$

Here the round brackets are used to denote symmetrization over the indices within.

It turns out that the most part of known interesting Killing forms satisfy for some constant c the additional equation

$$\nabla_X(d\psi) = cX^* \wedge \psi \,, \tag{2}$$

for any vector field X on M.

Definition 2. *The particular class of tensors which satisfy the above relation are called* special Killing forms [19].

We introduce below a symmetric generalization of the Killing vectors; if a symmetric tensor $K_{i_1\cdots i_r}$ of rank $r > 1$ satisfies the generalized Killing equation

$$\nabla_{(j}K_{i_1\cdots i_r)} = 0 \,,$$

then it is called a *Stäckel-Killing tensor.*

We remark that for any geodesic γ with tangent vector $\dot\gamma^i$

$$Q_K = K_{i_1\cdots i_r}\dot\gamma^{i_1}\cdots\dot\gamma^{i_r} \,, \tag{3}$$

is constant along γ.

This stands as an analogue of the conserved quantities associated with Killing vectors. Given two Killing-Yano tensors $\psi^{i_1,\dots,i_k}$ and $\sigma^{i_1,\dots,i_k}$ there is a Stäckel-Killing tensor of rank 2:

$$K^{(\psi,\sigma)}_{ij} = \psi_{ii_2\dots i_k}\sigma_j{}^{i_2\dots i_k} + \sigma_{ii_2\dots i_k}\psi_j{}^{i_2\dots i_k} \,. \tag{4}$$

This result represents an important connection between these two generalizations of the Killing vectors, and offers a method to generate higher order integrals of motion by identifying the complete set of Killing forms.

2.2 Special Killing forms on Sasaki-Einstein manifolds

We pass now to a remarkable class of manifolds where special Killing forms are known to exist; these are Sasaki-Einstein manifolds. In the following we give a brief presentation. First we introduce the metric cone $C(M)$ of the manifold M. This is in fact the product manifold $M \times \mathbb{R}_{>0}$, with $\dim C(M) = 2n+2$, endowed with the warped metric $g_{cone} := dr^2 + r^2 g$.

Definition 3. *A differentiable manifold M is called* Sasaki manifold *if its metric cone $C(M)$ is a Kähler manifold.*

If $\mathcal{J}$ represents the complex structure on the cone manifold, then we denote

$$\tilde{K} := \mathcal{J}(r\frac{\partial}{\partial r}).$$

Considering the restriction of $\tilde{K}$ to the submanifold determined by the condition $r = 1$ we obtain the Reeb vector field $\mathcal{B}$ on the Sasaki manifold M. The dual 1-form of $\mathcal{B}$ on M is denoted by η, and extending on the cone manifold $C(M)$ we obtain that the dual of $\tilde{K}$ is $r^2\eta$. Now the Kähler form ω can be expressed as

$$\omega = \frac{1}{2}d(r^2\eta) = \frac{1}{2}\mathrm{i}\partial\bar{\partial}r^2 .$$

where $\mathbf{i} := \sqrt{-1}$ and ∂ and $\bar{\partial}$ are the canonical operators associated to $\mathcal{J}$.

From here we easily see that $F := \frac{r^2}{4}$ is the Kähler potential.

Definition 4. *An* Einstein manifold *is a Riemannian manifold (M, g) satisfying the Einstein condition*

$$Ric_g = \lambda g\,, \tag{5}$$

for a real constant λ, where Ric_g denotes the Ricci tensor of g.

Einstein manifolds with $\lambda = 0$ are called *Ricci-flat manifolds*. Finally, a *Sasaki-Einstein manifold* is a Riemannian manifold (M, g) that is both Sasaki and Einstein. We note that in the case of Sasaki-Einstein manifolds one always has (5) with the Einstein constant $\lambda = 2n$. It turns out that a Sasaki manifold M is Einstein if and only if the metric cone $C(M)$ is Kähler Ricci-flat. Consequently, the metric cone of a Sasaki-Einstein manifolds will be a Kähler manifold with flat Ricci tensor, i.e. a Calabi-Yau manifold (see e.g. [6]).

Definition 5. *A* toric Sasaki manifold *M is a Sasaki manifold whose Kähler cone $C(M)$ is a toric Kähler manifold* [10].

For instance, a five-dimensional toric Sasaki-Einstein manifold is a Sasaki-Einstein manifold with three $U(1)$ isometries. Then, using symplectic geometry of the cone $C(M)$ one can introduce canonical coordinates and the Sasaki-Einstein structure can be described in terms of toric data together with a single function G, a symplectic potential (see e.g. [17]). Remarkable examples are represented by the spaces $Y^{p,q}$ with relative prime numbers p and q, which topologically are S^1-fibration over $S^2 \times S^2$.

Next, we discuss the existence of special Killing forms on Sasaki-Einstein manifolds. In this respect, a remarkable correspondence between special Killing forms defined on such manifold M and the parallel forms defined on the corresponding metric cone $C(M)$ was stated by Semmelman [19].

Theorem 1. [19] *A $p-$dimensional differential form Ψ is a special Killing form on M if and only if the corresponding form*

$$\Psi_{cone} := r^p dr \wedge \Psi + \frac{r^{p+1}}{p+1} d\Psi \,, \tag{6}$$

is parallel on $C(M)$.

Using the defining equation (2) we can show that on a five-dimensional Sasaki manifold M two special Killing forms can be directly written employing the $1-$form η [19]:

$$\Psi_1 = \eta \wedge d\eta, \quad \Psi_2 = \eta \wedge (d\eta)^2 \,. \tag{7}$$

Other two closed conformal Killing forms called $*$-Killing forms, are also obtained:

$$\Phi_1 = d\eta, \quad \Phi_2 = (d\eta)^2 \,. \tag{8}$$

In the case of the Calabi-Yau cone $C(M)$, with the method offered in [19] two additional special Killing forms on M can be extracted using parallel forms of the cone. We introduce now the *holomorphic complex volume form*.

Definition 6. *If* Vol *is the volume form on the metric cone, then the holomorphic volume form Ω is defined by the relation*

$$\mathrm{Vol} = \frac{1}{(n+1)!}\omega^{n+1} = \frac{i^{n+1}}{2^{n+1}}(-1)^{n(n+1)/2}\Omega \wedge \bar{\Omega} \,.$$

We recall now the following classical fact.

Theorem 2. [16, Chapter 17] *On Ricci flat Kähler manifolds the holomorphic complex form is parallel.*

Consequently, in our setting the holomorphic complex volume form Ω of $C(M)$ and its conjugate [19] are the two needed parallel forms.

2.3 Symplectic potential and complex coordinates

In order to write the holomorphic complex form we need complex coordinates and corresponding metric coefficients on the cone manifold. In the previous papers [25, 20] these coordinates were obtained by a direct computation. In order to show that this can be done using the classical Delzant construction and symplectic potential we start out by considering first the symplectic (action-angle) coordinates (y^i, Φ^i). The angular coordinates Φ^i will generate the toric action.

In order to obtain the y^i the key ingredient is represented by the momentum map $\mu = \frac{1}{2}r^2\eta$. We have the correspondence

$$y^i = \mu(\partial/\partial\Phi^i)\,. \tag{9}$$

The corresponding Kähler metric on the cone $C(M)$ is constructed using the *symplectic potential* G [15, 1].

For this purpose we briefly present below the Delzant result.

A *Delzant polytope* is a convex polytope such that there are n edges meeting at each vertex, each edge meeting at the vertex is of form $1+tu_i$, where $u_i \in \mathbb{Z}^n$, and $\{u_i\}$ can be chosen to form a basis in $\mathbb{Z}^n$. This polytope can be described by the inequalities

$$l_A(y) := \langle y, v_A\rangle \geq 0, \text{ for } 1 \leq A \leq d\,,$$

where $\{v_A\}$ are inward pointing normal vectors to the facets of the polytope and d is the number of facets [1, 10].

It is possible to associate to any Delzant polytope $P \in \mathbb{R}^n$ a close connected symplectic manifold M, together with a Hamiltonian $\mathbb{T}^n$ action on the manifold. In fact it can be shown that the polytope turns out to be the image of the momentum map, $P = \mu(M)$ (see e.g. [12]).

Remark 1. *In the case of the Calabi-Yau cone we take $C(M)$ to be Gorenstein which is a necessary condition to admit a Ricci-flat Kähler metric and M to admit a Sasaki-Einstein metric. For affine toric varieties it is well-known that $C(M)$ being Gorenstein is equivalent to the existence of a basis for the torus $\mathbb{T}^n$ for which*

$$v_a = (1, w_a)\,, \tag{10}$$

for each $a = 1, \cdots, d$ and $w_a \in \mathbb{Z}^{n-1}$ [14, 15].

The symplectic potential G can be written in terms of the toric data [15]

$$G = G^{can} + G^{\mathcal{B}} + h. \tag{11}$$

In the above description of G the first terms is the canonical symplectic potential and it is computed using the corresponding Delzant polytope.

$$G^{can} = \frac{1}{2}\sum_A l_A(y)\log l_A(y),$$

The second term is related to the Reeb vector field $\mathcal{B}$. Here we define the affine function $l_{\mathcal{B}} := \langle \mathcal{B}, \cdot\rangle$, and $l_\infty := \langle \sum_A v_A, \cdot\rangle$ and $G^{\mathcal{B}}$ is written as

$$G^{\mathcal{B}} = \frac{1}{2}l_{\mathcal{B}}(y)\log l_{\mathcal{B}}(y) - \frac{1}{2}l_\infty(y)\log l_\infty(y)\,.$$

Finally, as in the general case G needs to satisfy the Monge-Ampère equation, a homogeneous function h of degree 1 in variables y is added.

From equation (11) it becomes clear the relevance of the Reeb vector field $\mathcal{B}$ and the function h. Note that $\mathcal{B}$ is constant [15]. We make here some references regarding the possibility to calculate this vector. According to the AdS/CFT correspondence the volume of the Sasaki-Einstein space corresponds to the central charge of the dual conformal field theory. There are two known different algebraic procedures to extract the components of the Reeb vector from the toric data. The first procedure is based on *the maximization of the central charge* (a-maximization) [11] used in connection with the computation of the Weyl anomaly in 4-dimensional field theory. The second one is known as *volume minimization* (or Z-minimization) [15].

Using now the symplectic potential we can write the metric on the Kähler manifold with respect to the symplectic coordinates

$$ds^2 = G_{ij} dy^i dy^j + G^{ij} d\Phi^i d\Phi^j \ ,$$

where the metric coefficients are computed

$$G_{ij} = \frac{\partial^2 G}{\partial y^i \partial y^j} \ ,$$

with $\left(G^{ij}\right) := \left(G_{ij}\right)^{-1}$.

The main outcome of the above construction is that it allows us to pass to the complex coordinates $z^i := x^i + \mathrm{i}\Phi^i$.

Considering the Kähler potential F as the Legendre dual of G

$$F(x) = \left(y^i \frac{\partial G}{\partial y^i} - G \right) : (y = \partial F / \partial x) \ .$$

we can obtain the coordinates x^i using the Legendre transform

$$x^i = \frac{\partial G}{\partial y^i}, \ y^i = \frac{\partial F}{\partial x^i} \ .$$

The metric structure is now written with respect to the coordinate patch (x^i, Φ^i) in the following manner

$$ds^2 = F_{ij} dx^i dx^j + F_{ij} d\Phi^i d\Phi^j \ ,$$

where the metric coefficients are again obtained using the Hessian of the Kähler potential F, i.e.

$$F_{ij} = \frac{\partial^2 F}{\partial x^i \partial x^j} \ .$$

Another useful remark is [1]

$$\left(F_{ij}\right) = \left(G^{ij}\right) \ . \tag{12}$$

Using (11), we are able to express the coordinates x^i and the metric coefficients G_{ij}

$$x^i = \frac{\partial G}{\partial y^i} = \frac{1}{2}\sum_A v_A^i \log l_A(y) + \frac{1}{2}\mathcal{B}^i(1 + \log l_{\mathcal{B}}(y))$$
$$- \frac{1}{2}\sum_A v_A^i \log l_\infty(y) + \lambda_i \,,$$
$$G_{ij} = \frac{1}{2}\sum_A \frac{v_A^i v_A^j}{l_A(y)} + \frac{1}{2}\frac{\mathcal{B}_i\mathcal{B}_j}{l_{\mathcal{B}}(y)} - \frac{1}{2}\frac{\sum_A v_A^i \sum_A v_A^j}{l_\infty(y)} \,.$$

2.4 The holomorphic volume form in complex coordinates

As we can calculate the complex coordinates and the metric coefficients in the above manner, we can use them to express the holomorphic volume form.

Eventually ignoring the multiplicative constant, Ω can be written as [15]

$$\Omega = \exp(i\alpha) \det(F_{ij})^{1/2} dz^1 \wedge .. \wedge dz^n \,.$$

We use in the following the fact that the Calabi-Yau metric cone is Ricci flat. From the classical formula

$$\rho = -\mathrm{i}\partial\bar{\partial} \log \det(F_{ij}) \,,$$

using (12), by a simple computation we get (see [15])

$$\det(G_{ij}) = \exp\left(2\gamma_i \frac{\partial G}{\partial y^i} - c\right) , \tag{13}$$

with constants γ_i, and c.

Now, as the metric has to be smooth, from (13) it turns out that [17]

$$\gamma = (-1, 0, .., 0) \,,$$

and, consequently, the Hessian of the Kähler potential is written

$$\det(F_{ij}) = \exp\big(2x^1 + c\big).$$

Plugging this final result in the above formula, we get

$$\Omega = \exp(x^1 + i\alpha) dz^1 \wedge .. \wedge dz^n \,.$$

Finally, Ω should be closed (as it is parallel), and we can choose the phase α to be Φ^1. We obtain the desired result.

Proposition 1. [15] *With respect to the complex coordinates (z^i) the holomorphic volume form has the following simple form*

$$\Omega = \exp(z^1) dz^1 \wedge .. \wedge dz^n. \tag{14}$$

In the next sections we use this relation in order to extract the special Killing forms on manifolds of Sasaki-Einstein type.

3 Symplectic and complex coordinates on $Y^{p,q}$

The metric tensor of the 5-dimensional $Y^{p,q}$ manifold can be written as [14]

$$\begin{aligned} ds^2 = & \frac{1-c\,y}{6}(d\theta^2+\sin^2\theta\, d\phi^2)+\frac{1}{w(y)q(y)}dy^2+\frac{q(y)}{9}(d\psi-\cos\theta\, d\phi)^2 \\ & + w(y)\left[d\alpha+\frac{ac-2y+c\,y^2}{6(a-y^2)}[d\psi-\cos\theta\, d\phi]\right]^2 , \end{aligned} \tag{15}$$

where

$$\begin{aligned} w(y) &= \frac{2(a-y^2)}{1-cy} , \\ q(y) &= \frac{a-3y^2+2cy^3}{a-y^2} . \end{aligned} \tag{16}$$

This metric is Einstein with $Ric_g = 4g$ for all values of the constants a, c. Moreover the space is also Sasaki. For $c = 0$ the metric takes the local form of the standard homogeneous metric on $T^{1,1}$ [14]. Otherwise the constant c can be rescaled by a diffeomorphism and in what follows we take $c = 1$. For $0 < \alpha < 1$, we can take the range of the angular coordinates (θ, ϕ, ψ) to be $0 \leq \theta \leq 2\pi\,, 0 \leq \phi \leq 2\pi\,, 0 \leq \psi \leq 2\pi$. Choosing $0 < a < 1$ the roots y_i of the cubic equation

$$a - 3y^2 + 2y^3 = 0\,, \tag{17}$$

are real, one negative (y_1) and two positive (y_2, y_3). If the smallest of the positive roots is y_2, one can take the range of the coordinate y to be

$$y_1 \leq y \leq y_2\,.$$

We note that the parameter a of the cubic equation (17) can be expressed in terms of the two relatively prime positive integers p and q as

$$a = \frac{1}{2} - \frac{p^2-3q^2}{4p^3}\sqrt{4p^2-3q^2}, \tag{18}$$

and the roots of the cubic $a - 3y^2 + 2y^3$ are

$$y_1 = \frac{1}{4p}\left(2p-3q-\sqrt{4p^2-3q^2}\right), \tag{19}$$

$$y_2 = \frac{1}{4p}\left(2p+3q-\sqrt{4p^2-3q^2}\right), \tag{20}$$

$$y_3 = \frac{1}{2} + \frac{\sqrt{4p^2-3q^2}}{2p}. \tag{21}$$

We can now define (see [14]):

$$\alpha \equiv \ell\gamma\,, \tag{22}$$

with

$$\ell = \frac{q}{3q^2 - 2p^2 + p\sqrt{4p^2 - 3q^2}}\,, \tag{23}$$

and the following change of variables [14]

$$\alpha = -\beta/6 - \psi'/6\,,\ \psi = \psi'\,. \tag{24}$$

Therefore the metric (15) takes the local Sasaki-Einstein form

$$ds^2 = ds^2(B_4) + w(y)[d\alpha + A]^2,$$

where $B_4 = S^2 \times S^2$, $ds^2(B_4)$ is the non-trivial metric on B_4 described in [9] and A is a 1-form given by

$$A = f(y)(d\psi - \cos\theta d\phi)\,, \tag{25}$$

where

$$f(y) = \frac{a - 2y + y^2}{6(a - y^2)}\,. \tag{26}$$

Moreover, the Reeb vector $\mathcal{B}$ and the dual 1-form η are given by [14]:

$$\mathcal{B} = 3\frac{\partial}{\partial\psi'} = 3\frac{\partial}{\partial\psi} - \frac{1}{2\ell}\frac{\partial}{\partial\gamma} = 3\frac{\partial}{\partial\psi} - \frac{1}{2}\frac{\partial}{\partial\alpha} \tag{27}$$

and

$$\eta = -2y(d\alpha + A) + \frac{1}{3}q(y)(d\psi - \cos\theta d\phi)\,. \tag{28}$$

We remark now that the 1-form η can be written in a simple way as:

$$\begin{aligned} \eta &= -2yd\alpha + \frac{1-y}{3}(d\psi - \cos\theta d\phi) \\ &= -2y\ell d\gamma + \frac{1-y}{3}(d\psi - \cos\theta d\phi)\,, \end{aligned} \tag{29}$$

and it is easy to see that $\eta(\mathcal{B}) = 1$.

In relation to the angular variables ϕ, ψ, γ, the basis for an effectively acting $\mathbb{T}^3$ action is [14]

$$\begin{aligned} e_1 &= \frac{\partial}{\partial\phi} + \frac{\partial}{\partial\psi}\,, \\ e_2 &= \frac{\partial}{\partial\phi} - \frac{l}{2}\frac{\partial}{\partial\gamma}\,, \\ e_3 &= \frac{\partial}{\partial\gamma}\,, \end{aligned} \tag{30}$$

with $l = p - q$.

It is easy to see now that, considering this basis, the Reeb vector has the components

$$\mathcal{B} = \left(3, -3, -\frac{3}{2}(l + \frac{1}{3\ell})\right)\,. \tag{31}$$

If we write now the basis (30) in the following form

$$e_i = \frac{\partial}{\partial \Phi^i} \,, \tag{32}$$

then we obtain after some standard computations that

$$\begin{pmatrix} \frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial \psi} \\ \frac{\partial}{\partial \gamma} \end{pmatrix} = \begin{pmatrix} 0 & 1 & \frac{l}{2} \\ 1 & -1 & -\frac{l}{2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial \Phi^1} \\ \frac{\partial}{\partial \Phi^2} \\ \frac{\partial}{\partial \Phi^3} \end{pmatrix} \,, \tag{33}$$

and therefore we derive

$$\begin{aligned} \Phi^1 &= \psi \,, \\ \Phi^2 &= \phi - \psi \,, \\ \Phi^3 &= \frac{l}{2}\phi - \frac{l}{2}\psi + \gamma \,. \end{aligned} \tag{34}$$

Using now the correspondence between the symplectic (action-angle) coordinates (y^i, Φ^i) and the momentum map μ:

$$y^i = \mu(\partial/\partial \Phi^i) = \frac{r^2}{2}\eta(\partial/\partial\Phi^i) \,, \tag{35}$$

we deduce that, in this new basis (32), the momentum map becomes

$$\vec{y} = (y^1, y^2, y^3) = \left[\frac{r^2}{6}(1-y)(1-\cos\theta) \,,\, -\frac{r^2}{6}(1-y)\cos\theta + \frac{r^2}{2} l \ell y \,,\, -\ell r^2 y\right] . \tag{36}$$

Next, in order to introduce the complex coordinates on the Calabi-Yau cone $C(Y^{p,q})$, we need the symplectic potential G. First of all, let us consider the toric data for $Y^{p,q}$ [14, 17]):

$$v_1 = [1, -1, -p] \,,\, v_2 = [1, 0, 0] \,,\, v_3 = [1, -1, 0] \,,\, v_4 = [1, -2, -p+q] \,. \tag{37}$$

Now, from (11) we remark that the symplectic potential in the case of $Y^{p,q}$ contains the function h, in contradistinction to the case of the homogeneous Sasaki-Einstein manifold $T^{1,1}$ (see [21, 22]). However, it can be proved that one can derive a more simple expression [17]

$$G = \sum_{A=1}^{6} \frac{1}{2} \langle v_A, y \rangle \log \langle v_A, y \rangle \,, \tag{38}$$

by introducing two additional vectors v_5 and v_6 as follow:

$$\begin{aligned} v_5 &:= \mathcal{B} - v_1 - v_3 = \left(1, -1, -\frac{1}{2}p + \frac{3}{2}q - \frac{1}{2\ell}\right) \,, \\ v_6 &:= -v_2 - v_4 = (-2, 2, p-q) \,. \end{aligned} \tag{39}$$

Now we note that the complex coordinates are

$$\begin{aligned} z^1 =& x^1 + \mathrm{i}\psi\,, \\ z^2 =& x^2 + \mathrm{i}(\phi - \psi)\,, \\ z^3 =& x^3 + \mathrm{i}(\frac{l}{2}\phi - \frac{l}{2}\psi + \gamma)\,. \end{aligned} \tag{40}$$

In order to derive the expression of x^i we use:

$$x^i = \frac{\partial G}{\partial y^i} = \frac{1}{2}\sum_1^6 v_A^i \log\langle v_A, y\rangle + \frac{1}{2}\sum_1^6 v_A^i\,. \tag{41}$$

In the sequel, for the sake of simplicity we will ignore the additive constants. Using now (36), (37) and (39) in (41) and taking account of (18), (19), (20), (21) and (23), we derive after some long but relatively straightforward algebraic calculations:

$$\begin{aligned} x^1 =& 3\log r + \log\sin\theta + \frac{1}{2}\log\left(y^3 - \frac{3}{2}y^2 + \frac{a}{2}\right)\,, \\ x^2 =& -3\log r - 2\log\cos\frac{\theta}{2} - \frac{1}{2}\log\left(y^3 - \frac{3}{2}y^2 + \frac{a}{2}\right)\,, \\ x^3 =& \frac{p(y_1 - y_3)}{1 - y_1}\log r - l\log\cos\frac{\theta}{2} \\ &+ \frac{p(1 - y_3)}{2(1 - y_1)}\log(y - y_3) - \frac{p}{2}\log(y - y_1)\,, \end{aligned} \tag{42}$$

Hence we can introduce on $C(Y^{p,q})$ the following patch of complex coordinates

$$\begin{aligned} z^1 =& \log\left(r^3\sin\theta\sqrt{y^3 - \frac{3}{2}y^2 + \frac{a}{2}}\right) + \mathrm{i}\psi\,, \\ z^2 =& -\log\left(r^3\cos^2\frac{\theta}{2}\sqrt{y^3 - \frac{3}{2}y^2 + \frac{a}{2}}\right) + \mathrm{i}(\phi - \psi)\,, \\ z^3 =& \log\frac{r^{\frac{p(y_1 - y_3)}{1 - y_1}}(y - y_3)^{\frac{p(1 - y_3)}{2(1 - y_1)}}}{\left(\cos\frac{\theta}{2}\right)^l\sqrt{(y - y_1)^p}} + \mathrm{i}\left(\frac{l}{2}\phi - \frac{l}{2}\psi + \gamma\right)\,. \end{aligned} \tag{43}$$

4 Special Killing forms on $Y^{p,q}$

In this section we will prove that the patch of complex coordinates (43) obtained above are the perfect ingredient in order to extract the special Killing forms on $Y^{p,q}$.

From (43), we obtain

$$\begin{aligned}
\exp(z^1) &= r^3 \sin\theta \sqrt{y^3 - \frac{3}{2}y^2 + \frac{a}{2}}\, \exp(\mathrm{i}\psi)\,, \\
dz^1 &= \frac{3}{r}dr + T_1, \\
dz^2 &= -\frac{3}{r}dr + T_2\,, \\
dz^3 &= \frac{p(y_1 - y_3)}{r(1-y_1)}dr + T_3\,.
\end{aligned}$$

where

$$\begin{aligned}
T_1 &:= \cot\theta d\theta + \frac{1}{2}\frac{3y^2 - 3y}{y^3 - \frac{3}{2}y^2 + \frac{a}{2}}dy + \mathrm{i}d\psi, \\
T_2 &:= \tan\frac{\theta}{2}d\theta - \frac{1}{2}\frac{3y^2 - 2y}{y^3 - \frac{3}{2}y^2 + \frac{a}{2}}dy + \mathrm{i}(d\phi - d\psi), \\
T_3 &:= \frac{l}{2}\tan\frac{\theta}{2}d\theta + \frac{p(y_1 - y_3)(y-1)}{2(1-y_1)(y-y_1)(y-y_3)}dy \\
&\quad + \mathrm{i}\left(\frac{l}{2}d\phi - \frac{l}{2}d\psi + d\gamma\right)\,.
\end{aligned} \tag{44}$$

Therefore, the holomorphic volume form is

$$\begin{aligned}
\Omega = \exp(z^1)dz^1 \wedge dz^2 \wedge dz^3 &= r^3 \sin\theta \sqrt{y^3 - \frac{3}{2}y^2 + \frac{a}{2}}\, \exp(\mathrm{i}\psi) \\
&\times \left(\frac{3}{r}dr + T_1\right) \wedge \left(-\frac{3}{r}dr + T_2\right) \wedge \left(\frac{p(y_1 - y_3)}{r(1-y_1)}dr + T_3\right)\,.
\end{aligned}$$

In our particular framework the equation (6) becomes

$$\Omega = r^2 dr \wedge \Psi + \frac{r^3}{3}d\Psi\,.$$

In order to extract Ψ we have to keep the trace of the differential form dr in the above equation. We derive

$$\Psi = \sin\theta \sqrt{y^3 - \frac{3}{2}y^2 + \frac{a}{2}}\, \exp(\mathrm{i}\psi)\left(3T_2 \wedge T_3 + 3T_1 \wedge T_3 + \frac{p(y_1 - y_3)}{1-y_1}T_1 \wedge T_2\right)\,. \tag{45}$$

Next, we will compute the wedge products from (45) using (44). After some

long but standard computations, we obtain

$$
\begin{aligned}
T_2 \wedge T_3 = \tan\frac{\theta}{2}&\left[\frac{p(y_1-y_3)(y-1)}{2(1-y_1)(y-y_1)(y-y_3)} + \frac{l}{4}\frac{3y(y-1)}{y^3-\frac{3}{2}y^2+\frac{a}{2}}\right] d\theta\wedge dy \\
&- \mathrm{i}\left[\frac{p(y_1-y_3)(y-1)}{2(1-y_1)(y-y_1)(y-y_3)} + \frac{l}{4}\frac{3y(y-1)}{y^3-\frac{3}{2}y^2+\frac{a}{2}}\right] dy\wedge d\phi \\
&+ \mathrm{i}\left[\frac{p(y_1-y_3)(y-1)}{2(1-y_1)(y-y_1)(y-y_3)} + \frac{l}{4}\frac{3y(y-1)}{y^3-\frac{3}{2}y^2+\frac{a}{2}}\right] dy\wedge d\psi \\
&- \frac{\mathrm{i}}{2}\frac{3y(y-1)}{y^3-\frac{3}{2}y^2+\frac{a}{2}} dy\wedge d\gamma + \mathrm{i}\tan\frac{\theta}{2} d\theta\wedge d\gamma \\
&- d\phi\wedge d\gamma + d\psi\wedge d\gamma\,,
\end{aligned}
\tag{46}
$$

$$
\begin{aligned}
T_1 \wedge T_3 = &\left[\frac{p(y_1-y_3)(y-1)}{2(1-y_1)(y-y_1)(y-y_3)}\cot\theta - \frac{l}{4}\frac{3y(y-1)}{y^3-\frac{3}{2}y^2+\frac{a}{2}}\tan\frac{\theta}{2}\right] d\theta\wedge dy \\
&- \mathrm{i}\left[\frac{p(y_1-y_3)(y-1)}{2(1-y_1)(y-y_1)(y-y_3)} + \frac{l}{4}\frac{3y(y-1)}{y^3-\frac{3}{2}y^2+\frac{a}{2}}\right] dy\wedge d\psi \\
&+ \mathrm{i}\frac{l}{2}\cot\theta d\theta\wedge d\phi + \mathrm{i}\cot\theta d\theta\wedge d\gamma - \mathrm{i}\frac{l}{2\sin\theta} d\theta\wedge d\psi \\
&+ \mathrm{i}\frac{l}{4}\frac{3y(y-1)}{y^3-\frac{3}{2}y^2+\frac{a}{2}} dy\wedge d\phi + \frac{\mathrm{i}}{2}\frac{3y(y-1)}{y^3-\frac{3}{2}y^2+\frac{a}{2}} dy\wedge d\gamma \\
&- \frac{l}{2} d\psi\wedge d\phi - d\psi\wedge d\gamma\,,
\end{aligned}
\tag{47}
$$

and

$$
\begin{aligned}
T_1 \wedge T_2 = &-\frac{1}{2\sin\theta}\frac{3y(y-1)}{y^3-\frac{3}{2}y^2+\frac{a}{2}} d\theta\wedge dy + \mathrm{i}\cot\theta d\theta\wedge d\phi \\
&- \mathrm{i}\frac{1}{\sin\theta} d\theta\wedge d\psi + \frac{\mathrm{i}}{2}\frac{3y(y-1)}{y^3-\frac{3}{2}y^2+\frac{a}{2}} dy\wedge d\phi - d\psi\wedge d\phi\,.
\end{aligned}
\tag{48}
$$

Introducing now (46)-(48) in (45) and ignoring the multiplicative constants, we derive

$$
\begin{aligned}
\Psi = &\sqrt{y^3-\frac{3}{2}y^2+\frac{a}{2}}\exp(\mathrm{i}\psi) \\
&\times\left[\left(a(y)d\theta\wedge dy + \frac{1}{2\ell}\sin\theta d\psi\wedge d\phi - 3\sin\theta d\phi\wedge d\gamma\right)\right. \\
&\left.+\mathrm{i}\left(\frac{1}{2\ell}d\theta\wedge d\psi + 3d\theta\wedge d\gamma - a(y)\sin\theta dy\wedge d\phi - \frac{1}{2\ell}\cos\theta d\theta\wedge d\phi\right)\right]\,,
\end{aligned}
\tag{49}
$$

where

$$\begin{aligned}
a(y) &= \frac{3p(y_1-y_3)(y-1)}{2(1-y_1)(y-y_1)(y-y_3)} - \frac{p(y_1-y_3)}{2(1-y_1)}\frac{3y(y-1)}{y^3-\frac{3}{2}y^2+\frac{a}{2}} \\
&= \frac{3py_2(y_1-y_3)}{1-y_1}\frac{1-y}{2y^3-3y^2+a} \\
&= -\frac{3}{2\ell}\frac{1-y}{2y^3-3y^2+a}.
\end{aligned}$$

Now, we can easily obtain the real special Killing forms computing the real and imaginary part of Ψ:

$$\begin{aligned}
\Re\Psi =& \sqrt{y^3-\frac{3}{2}y^2+\frac{a}{2}}\left[\cos\psi\left(a(y)d\theta\wedge dy+\frac{1}{2\ell}\sin\theta d\psi\wedge d\phi-3\sin\theta d\phi\wedge d\gamma\right)\right. \\
&\left.-\sin\psi\left(\frac{1}{2\ell}d\theta\wedge d\psi+3d\theta\wedge d\gamma-a(y)\sin\theta dy\wedge d\phi-\frac{1}{2\ell}\cos\theta d\theta\wedge d\phi\right)\right],
\end{aligned}$$

$$\begin{aligned}
\Im\Psi =& \sqrt{y^3-\frac{3}{2}y^2+\frac{a}{2}}\left[\sin\psi\left(a(y)d\theta\wedge dy+\frac{1}{2\ell}\sin\theta d\psi\wedge d\phi-3\sin\theta d\phi\wedge d\gamma\right)\right. \\
&\left.+\cos\psi\left(\frac{1}{2\ell}d\theta\wedge d\psi+3d\theta\wedge d\gamma-a(y)\sin\theta dy\wedge d\phi-\frac{1}{2\ell}\cos\theta d\theta\wedge d\phi\right)\right].
\end{aligned}$$

Next we will prove that the above special Killing forms agree with the special Killing forms Ξ and Υ recently obtained in [20, 25]. Indeed, if we denote

$$p(y) = \frac{2y^3-3y^2+a}{3(1-y)},$$

then we can easily remark that

$$a(y)p(y) = -\frac{1}{2\ell} \tag{50}$$

and

$$\sqrt{y^3-\frac{3}{2}y^2+\frac{a}{2}}a(y) = -\frac{3}{2\ell}\sqrt{\frac{1-y}{6p\,(y)}}. \tag{51}$$

Using now (22), (50) and (51), we obtain that the real special Killing forms $\Re\Psi$ and $\Im\Psi$ can be rewritten as:

$$\begin{aligned}
\Re\Psi =& -\frac{3}{2\ell}\sqrt{\frac{1-y}{6p\,(y)}}\left[\cos\psi\left(d\theta\wedge dy-p(y)\sin\theta d\psi\wedge d\phi+6p(y)\sin\theta d\phi\wedge d\alpha\right)\right. \\
&\left.-\sin\psi\left(-p(y)d\theta\wedge d\psi-6p(y)d\theta\wedge d\alpha-\sin\theta dy\wedge d\phi+p(y)\cos\theta d\theta\wedge d\phi\right)\right],
\end{aligned} \tag{52}$$

$$\begin{aligned}
\Im\Psi =& -\frac{3}{2\ell}\sqrt{\frac{1-y}{6p\,(y)}}\left[\sin\psi\left(d\theta\wedge dy-p(y)\sin\theta d\psi\wedge d\phi+6p(y)\sin\theta d\phi\wedge d\alpha\right)\right. \\
&\left.+\cos\psi\left(-p(y)d\theta\wedge d\psi-6p(y)d\theta\wedge d\alpha-\sin\theta dy\wedge d\phi+p(y)\cos\theta d\theta\wedge d\phi\right)\right].
\end{aligned} \tag{53}$$

Therefore, because $\Re\Psi$ and $\Im\Psi$ coincide with Ξ and Υ modulo a multiplicative constant, we conclude that indeed the special Killing forms obtained in this article agree with the results previously obtained in [20, 25] with different approaches.

5 Conserved quantities and complete integrability of geodesic motion in $Y^{p,q}$ spaces

On a manifold with coordinates x^μ and metric $g_{\mu\nu}$ the geodesics can be defined as the trajectories of test-particles with proper-time Hamiltonian

$$H = \frac{1}{2} g^{\mu\nu} P_\mu P_\nu \,. \tag{54}$$

Here P_μ are canonical momenta conjugate to the coordinates x^μ, $P_\mu = g_{\mu\nu}\dot{x}^\nu$ with overdot denoting proper time derivative.

The system of a free particle admits conserved quantities (3) which commute with the Hamiltonian (54) in the sense of Poisson brackets:

$$\{K, H\} = 0 \,. \tag{55}$$

Let us recall that in classical mechanics a Hamiltonian system with Hamiltonian H (54) and integrals of motion K_j is called *completely integrable* (or Liouville integrable) if it allows n integrals of motion $H, K_1, \ldots, K_{n-1}$ which are well-defined functions on the phase space, in involution

$$\{H, K_j\} = 0 \,, \quad \{K_j, K_k\} = 0 \,, \quad j,k = 1, \cdots, n-1 \,, \tag{56}$$

and functionally independent. A system is *superintegrable* if it is completely integrable and allows further functionally independent integrals of motion.

For $Y^{p,q}$ spaces the conjugate momenta to the coordinates $(\theta, \phi, y, \alpha, \psi)$ are [5]:

$$\begin{aligned}
P_\theta &= \frac{1-y}{6}\dot{\theta} \,, \\
P_\phi + \cos\theta P_\psi &= \frac{1-y}{6}\sin^2\theta\dot{\phi} \,, \\
P_y &= \frac{1}{6p(y)}\dot{y} \,, \\
P_\alpha &= w(y)\left(\dot{\alpha} + f(y)\left(\dot{\psi} - \cos\theta\dot{\phi}\right)\right) \,, \\
P_\psi &= w(y)f(y)\dot{\alpha} + \left[\frac{q(y)}{9} + w(y)f^2(y)\right]\left(\dot{\psi} - \cos\theta\dot{\phi}\right) \,.
\end{aligned} \tag{57}$$

From the isometry $SU(2)\times U(1)\times U(1)$ of the metric (15) we have that the momenta P_ϕ, P_ψ and P_α are conserved. P_ϕ is the third component of the $SU(2)$

angular momentum and P_ψ, P_α are associated to the $U(1)$ factors. In addition, the total $SU(2)$ angular momentum

$$\vec{J}^{\,2} = P_\theta^2 + \frac{1}{\sin^2\theta}\left(P_\phi + \cos\theta P_\psi\right)^2 + P_\psi^2 \tag{58}$$

is also conserved [5, 18].

The next conserved quantities, quadratic in momenta, will be expressed in terms of Stäckel-Killing tensors as in (3). The Stäckel-Killing tensors of rank two on $Y^{p,q}$ will be constructed from Killing-Yano tensors according to (4). For this purpose we shall use the Killing-Yano tensor Ψ_1 from (7) and the additional parallel forms of degree 2, associated with the real and imaginary parts of the holomorphic $(3,0)$ volume form Ω of the cone $C(Y^{p,q})$.

The first Stäckel-Killing tensor $K^{(1)}_{\mu\nu}$ is constructed according to (4) using the real part of the Killing form Ψ (52):

$$K^{(1)}_{\mu\nu} = (\Re\Psi)_{\mu\lambda}(\Re\Psi)^{\lambda}{}_{\nu}\,. \tag{59}$$

The corresponding conserved quantity (3)[1], modulo a multiplicative constant, is [2]

$$\begin{aligned}K^{(1)} =& 6(1-y)\dot{\theta}\dot{\theta} + \frac{3+a-6y+2y^3+(-3+a+6y-6y^2+2y^3)\cos 2\theta}{1-y}\dot{\phi}\dot{\phi}\\ &- 24\frac{(a+(-3+2y)y^2)\cos\theta}{1-y}\dot{\phi}\dot{\alpha} - 4\frac{(a+(-3+2y)y^2)\cos\theta}{1-y}\dot{\phi}\dot{\psi}\\ &+ 18\frac{1-y}{a+(-3+2y)y^2}\dot{y}\dot{y} + 72\frac{a+(-3+2y)y^2}{1-y}\dot{\alpha}\dot{\alpha}\\ &+ 24\frac{a+(-3+2y)y^2}{1-y}\dot{\alpha}\dot{\psi} + 2\frac{a+(-3+2y)y^2}{1-y}\dot{\psi}\dot{\psi}\,.\end{aligned} \tag{60}$$

The next Stäckel-Killing tensor will be constructed from the imaginary part of Ψ (53):

$$K^{(2)}_{\mu\nu} = (\Im\Psi)_{\mu\lambda}(\Im\Psi)^{\lambda}{}_{\nu}\,, \tag{61}$$

and we find that this tensor produces the same conserved quantity $K^{(1)}$ (60).

The mixed combination of $\Re\Psi$ and $\Im\Psi$ produces the Stäckel-Killing tensor

$$K^{(3)}_{\mu\nu} = (\Re\Psi)_{\mu\lambda}(\Im\Psi)^{\lambda}{}_{\nu} + (\Im\Psi)_{\mu\lambda}(\Re\Psi)^{\lambda}{}_{\nu}\,, \tag{62}$$

but it proves that all components of this tensor vanish.

Finally we construct the Stäckel-Killing tensor from the Killing form Ψ_1:

$$K^{(4)}_{\mu\nu} = (\Psi_1)_{\mu\lambda\sigma}(\Psi_1)^{\lambda\sigma}{}_{\nu}\,. \tag{63}$$

[1] In [18] the expression of this conserved quantity has some misprints. Consequently, the evaluation of the number of functionally independent set of integrals of motion is affected and the system is not superintegrable.

From (7) using the 1-form η (29) we get

$$\begin{aligned}\Psi_1 = &(1-y)^2 \sin\theta \, d\theta \wedge d\phi \wedge d\psi - 6 dy \wedge d\alpha \wedge d\psi \\ &+ 6\cos\theta \, d\phi \wedge dy \wedge d\alpha - 6(1-y)y \sin\theta \, d\theta \wedge d\phi \wedge d\alpha \,.\end{aligned} \tag{64}$$

and the corresponding conserved quantity, modulo a multiplicative constant, is [2]

$$\begin{aligned}K^{(4)} = &\, 6(1-y)\dot\theta\dot\theta - 24\frac{(a+(-4+5y-2y^2)y)\cos\theta}{1-y}\dot\phi\dot\alpha \\ &+ \frac{7+a-18y+12y^2-2y^3+(1+a-6y+6y^2-2y^3)\cos 2\theta}{1-y}\dot\phi\dot\phi \\ &- 4\frac{(a-(2-y)^2(-1+2y))\cos\theta}{1-y}\dot\phi\dot\psi \\ &+ 18\frac{1-y}{a+(-3+2y)y^2}\dot y\dot y + 72\frac{a+(1-2y)y^2}{1-y}\dot\alpha\dot\alpha \\ &+ 24\frac{a+(-4+5y-2y^2)y}{1-y}\dot\alpha\dot\psi + 2\frac{a-(2-y)^2(-1+2y)}{1-y}\dot\psi\dot\psi \,.\end{aligned} \tag{65}$$

Having in mind that $K^{(1)} = K^{(2)}$ and $K^{(3)}$ vanishes, we shall verify if the set $H, P_\phi, P_\psi, P_\alpha, \vec{J}^{\,2}, K^{(1)}, K^{(4)}$ constitutes a functionally independent set of constants of motion for the geodesics of $Y^{p,q}$ constructing the Jacobian:

$$\mathcal{J} = \frac{\partial(H, P_\phi, P_\psi, P_\alpha, \vec{J}^{\,2}, K^{(1)}, K^{(4)})}{\partial(\theta, \phi, y, \alpha, \psi, \dot\theta, \dot\phi, \dot y, \dot\alpha, \dot\psi)} \,. \tag{66}$$

The rank of this Jacobian is 5, exactly the number of the degrees of freedom, which means that the system is completely integrable. In spite of the presence of the Stäckel-Killing tensors $K^{(1)}$ and $K^{(4)}$, the system is not superintegrable, $K^{(1)}$ and $K^{(4)}$ being a combination of the first integrals $H, P_\phi, P_\psi, P_\alpha, \vec{J}^{\,2}$. It is interesting to note that the toric Sasaki-Einstein spaces $Y^{p,q}$ spaces possess several Killing-Yano tensors, but these Killing forms do not generate new Stäckel-Killing tensors, i.e. genuine conserved quantities.

6 Conclusions

In this article we investigate the complex structure of the conifold $C(Y^{p,q})$ basically making use of the interplay between symplectic and complex approaches of the Kähler toric manifolds. The description of the Calabi-Yau manifold $C(Y^{p,q})$ with toric data allows us to write explicitly the complex coordinates and special Killing forms. Using the complete set of Killing vectors and Stäckel-Killing tensors on $Y^{p,q}$ we construct the corresponding conserved quantities and proved the complete integrability of geodesic motion.

The present investigation is important in the context of the AdS/CGT correspondence. By focusing on the geodesics on the Sasaki-Einstein spaces, the

paper refers to geometries produced by D-branes on non-flat bases going towards the general goal of classifying all supersymmetric geometries with integrable geodesics. It is quite remarkable the fact that while the point-like strings (geodesic) equations are integrable in some backgrounds, the corresponding extended classical string motion is not integrable in general [3, 4, 24, 7, 8].

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